Gravitational 2-body problem : II

- bound vs. unbound orbits
- elliptical shape of bound orbits
- period of elliptical orbits: Kepler's laws

Without loss of generality (choice of coordinates) we set $\theta_0 = 0$ in our orbit formula:

$$u(\theta) = \frac{1}{r(\theta)} = \frac{1}{r_0}\left(1 + \epsilon \cos\theta\right)$$

The eccentricity ϵ is a dimensionless ~~, non-negative~~ parameter, where $\epsilon = 0$ corresponds to a circular orbit. Since the sign of ϵ can be flipped by the rotation $\theta \to \theta + \pi$, we may assume $\epsilon \geq 0$.

It is interesting to see how the total energy

$$E = \underbrace{\frac{L_z^2}{2\mu}}_{\frac{Ar_0}{2}} \left[\left(\frac{du}{d\theta} \right)^2 + u^2 \right] - Au$$

depends on the parameters r_0 and ϵ of our orbit:

$$E = \frac{L_z^2}{2\mu} \left[\frac{\epsilon^2}{r_0^2} \sin^2\theta + \frac{1}{r_0^2} + \cancel{\frac{2\epsilon}{r_0^2} \cos\theta} + \frac{\epsilon^2}{r_0^2} \cos^2\theta \right] - \frac{A}{r_0} (1 + \cancel{\epsilon\cos\theta})$$

$$= \frac{Ar_0}{2} \left[\frac{\epsilon^2}{r_0^2} + \frac{1}{r_0^2} \right] - \frac{A}{r_0}$$

$$= \frac{A}{r_0} \left[\frac{\epsilon^2}{2} - \frac{1}{2} \right] = \frac{A}{2r_0} (\epsilon^2 - 1)$$

We see that if $0 \le \epsilon \le 1$ the energy E is negative, and inconsistent with the scenario where the separation between the masses grows without bound (since then the energy would be purely kinetic and positive). The case $0 \le \epsilon < 1$ therefore corresponds to bound orbits.

A simple calculation shows the shape of a bound orbit is an ellipse. From the orbit equation

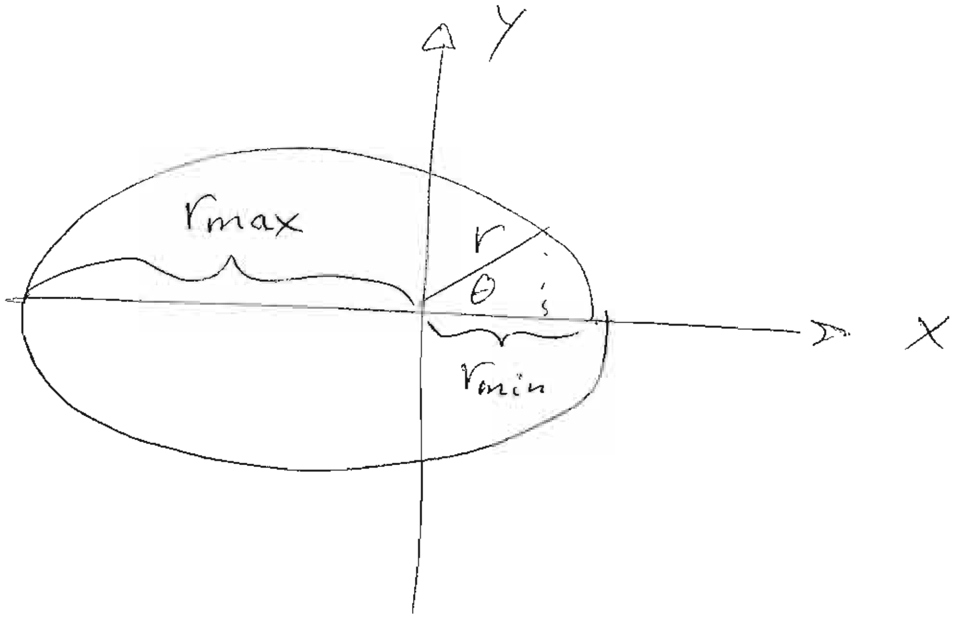
$$\frac{1}{r(\theta)} = \frac{1}{r_0}(1 + \epsilon \cos\theta)$$

we see that when $\epsilon < 1$ the

separation $r(\theta)$ varies between the extremes

$$r_{max} = \frac{r_0}{1-\epsilon}, \quad r_{min} = \frac{r_0}{1+\epsilon}.$$

$$(\theta = \pi) \qquad\qquad (\theta = 0)$$



From $\quad r_0 = r + r\epsilon\cos\theta$

$$= r + \epsilon x$$

we find

$$(r_0 - \epsilon x)^2 = r^2 = x^2 + y^2$$

$$r_0^2 = (1-\epsilon^2)x^2 + 2\epsilon r_0 x + y^2$$

$$(1-\epsilon^2)(x-x_0)^2+y^2 = \underbrace{r_0^2+(1-\epsilon^2)x_0^2}_{\left(1+\frac{\epsilon^2}{1-\epsilon^2}\right)r_0^2=\frac{r_0^2}{1-\epsilon^2}}$$

$$x_0 = -\frac{\epsilon}{1-\epsilon^2}r_0$$

define: $a=\dfrac{r_0}{1-\epsilon^2}$, $b=\dfrac{r_0}{\sqrt{1-\epsilon^2}}$

and x, y satisfy:

$$\frac{(x-x_0)^2}{a^2}+\frac{y^2}{b^2}=1,$$

the equation of an ellipse with semi-major axis a, semi-minor axis b. As $\epsilon \to 1$ the ellipse becomes very elongated:

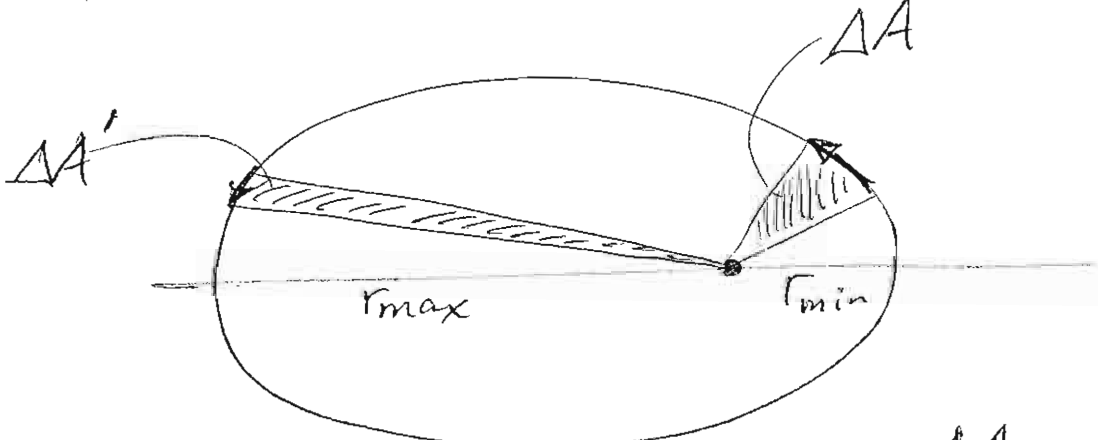
$$\frac{a}{b}=\frac{1}{\sqrt{1-\epsilon^2}} \longrightarrow \infty$$

In that same limit

$$r_{min} \longrightarrow \frac{r_0}{2} \;, \; r_{max} \longrightarrow \infty$$

and the orbit becomes <u>unbound</u> at $\epsilon = 1$. We consider the case $\epsilon > 1$ later.

Kepler's 1st law states that the "position vector sweeps out equal areas in equal time". This actually refers to the <u>relative</u> position vector:



$$\Delta A = \Delta A'$$

This law is actually a direct consequence of the constancy of L_z :

$$\Delta A = \left(\frac{\Delta\theta}{2\pi}\right)\pi r^2 = \frac{1}{2} r^2 \dot{\theta}\, \Delta t$$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \dot{\theta} = \frac{L_z}{2\mu} = \text{constant}$$

Kepler's 2nd law :

$$(\text{orbit period})^2 \propto \left(\begin{matrix}\text{semi-major}\\ \text{axis}\end{matrix}\right)^3$$

This follows from the integral of the area derivative over one period :

$$A = \oint_0^T \dot{A}\, dt = \frac{L_z}{2\mu} \cdot T$$

Geometrically,

$$A = \pi a b$$

But we will express this in terms of just a:

$$b^2 = \frac{\cancel{r_0^2}}{1-\epsilon^2} = a^2(1-\epsilon^2)$$

$$A^2 = \pi^2 a^2 \cdot a^2(1-\epsilon^2)$$

$$= \pi^2 a^3 r_0$$

Comparing the two expressions for A^2:

$$\frac{L_z^2}{(2\mu)^2} T^2 = \pi^2 a^3 r_0$$

$$= \pi^2 a^3 \frac{L_z^2}{A\mu} \quad \text{(previous lecture)}$$

$$\Rightarrow T^2 = 4\pi^2 \left(\frac{\mu}{A}\right) a^3$$

$$= \left(4\pi^2 / G(M_1+M_2)\right) a^3$$

It is useful to know that this law applies to arbitrary elliptic orbits, not just nearly circular ones.